

On quasi-normal modes, area quantization and Bohr correspondence principle

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Abstract

In Int. Journ. Mod. Phys. D 14, 181 (2005), the author Khriplovich verbatim claims that “the correspondence principle does not dictate any relation between the asymptotics of quasinormal modes and the spectrum of quantized black holes” and that “this belief is in conflict with simple physical arguments”. In this paper we analyze Khriplovich’s criticisms and realize that they work only for the original proposal by Hod, while they do not work for the improvements suggested by Maggiore and recently finalized by the author and collaborators through a connection between Hawking radiation and black hole (BH) quasi-normal modes (QNMs). This is a model of quantum BH somewhat similar to the historical semi-classical model of the structure of a hydrogen atom introduced by Bohr in 1913. Thus, QNMs can be really interpreted as BH quantum levels (the “electrons” of the “Bohr-like BH model”).

Our results have also important implications on the BH information puzzle.

1 Introduction

BH QNMs are frequencies of radial spin $j = 0, 1, 2$ for scalar, vector and gravitational perturbation respectively, obeying a time independent Schrödinger-like equation [1, 2]. Such BH modes of energy dissipation have a frequency which is allowed to be complex [1, 2]. In a remarkable paper [3], York proposed the intriguing idea to model the quantum BH in terms of BH QNMs. More recently, by using Bohr’s Correspondence Principle, Hod proposed that QNMs should

release information about the area quantization as QNMs are associated to absorption of particles [4, 5]. Hod's work was improved by Maggiore [6] who solved some important problems. On the other hand, as QNMs are countable frequencies, ideas on the continuous character of Hawking radiation did not agree with attempts to interpret QNMs in terms of emitted quanta, preventing to associate QNMs modes to Hawking radiation [1]. Recently, Zhang, Cai, Zhan and You [7, 8, 9, 10] and the author and collaborators [11, 12, 13, 14] observed that the non-thermal spectrum by Parikh and Wilczek [21, 22] also implies the countable character of subsequent emissions of Hawking quanta. This issue enables a natural correspondence between QNMs and Hawking radiation, permitting to interpret QNMs also in terms of emitted energies [11, 12, 13, 14]. The model that has been developed in [11, 12, 13, 14] is a BH model somewhat similar to the historical semi-classical model of the structure of a hydrogen atom introduced by Bohr in 1913 [17, 18]. In fact, in the Bohr-like model for BHs in [11, 12, 13, 14] QNMs represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies represent the energy "shells". The emission of Hawking quanta and the absorptions of particles represent, in turn, the jumps among the various quantum levels. The Bohr-like model for BHs has also important implications on the BH information paradox [25].

2 Hod's original proposal

For Schwarzschild BH and in strictly thermal approximation, QNMs are usually labelled as ω_{nl} , where n and l are the "overtone" and the angular momentum quantum numbers [1, 2, 4, 5, 6]. For each $l \geq 2$ for BH perturbations, we have a countable infinity of QNMs, labelled by n ($n = 1, 2, \dots$) [1, 2, 4, 5, 6]. Working with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units), for large n BH QNMs become independent of l having the structure [1, 2, 4, 5, 6]

$$\begin{aligned}\omega_n &= \ln 3 \times T_H + 2\pi i(n + \frac{1}{2}) \times T_H + \mathcal{O}(n^{-\frac{1}{2}}) = \\ &= \frac{\ln 3}{8\pi M} + \frac{2\pi i}{8\pi M}(n + \frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}).\end{aligned}\tag{1}$$

Although the behavior (1) is independent of the orbital angular momentum l the detailed values are irregular and l -dependent for non very large imaginary part of ω_n [1]. The number of quasi-normal frequencies is infinite and the real part in (1) equals [1]

$$\frac{\ln 3}{4\pi} \approx 0.087424.\tag{2}$$

Hod was the first author to note that the constant (2) can be written in terms of $\ln 3$ and proposed an intriguing interpretation in order to explain this fact [4, 5], as we will see below. On the other hand, the imaginary part of (1) can be easily understood [2]. The quasi-normal frequencies determine the position of poles of a Green's function on the given background and the Euclidean BH solution

converges to a thermal circle at infinity with the inverse temperature $\beta_H = \frac{1}{T_H}$ [2]. Thus, the spacing of the poles in eq. (1) coincides with the spacing $2\pi i T_H$ expected for a thermal Green's function [2].

In a famous paper, Bekenstein [15] showed that the area quantum of the Schwarzschild BH is $\Delta A = 8\pi$ (we recall that the *Planck distance* $l_p = 1.616 \times 10^{-33}$ cm is equal to one in Planck units). Using properties of the spectrum of Schwarzschild BH QNMs a different numerical coefficient has been found by Hod in [4, 5]. Hod's analysis started by observing that, as for the Schwarzschild BH the *horizon area* A is related to the mass through the relation $A = 16\pi M^2$, a variation ΔM in the mass generates a variation

$$\Delta A = 32\pi M \Delta M \quad (3)$$

in the area. By considering a transition from an unexcited BH to a BH with very large n , Hod [4, 5] assumed *Bohr's correspondence principle* (which states that transition frequencies at large quantum numbers should equal classical oscillation frequencies) [16, 17, 18] to be valid for large n and enabled a semi-classical description even in absence of a complete theory of quantum gravity. In his approach, Hod [4, 5] assumed that the relevant frequencies were the real part of the frequencies (1). Hence, the minimum quantum which can be absorbed in the transition is [4, 5]

$$\Delta M = \omega = \frac{\ln 3}{8\pi M}. \quad (4)$$

This gives $\Delta A = 4 \ln 3$. This results was different from the original result of Bekenstein [15] while the presence of the numerical factor $4 \ln 3$ stimulated possible connections with loop quantum gravity [19].

3 Criticisms by Khriplovich

Hod's approach has been criticized by Khriplovich [20], who claims that properties of ringing frequencies cannot be related directly to Bohr correspondence principle. Here we show that this criticism works only for the original proposal by Hod [4, 5], while it does not work for the improvements suggested by Maggiora [6] and recently finalized by the author and collaborators [11, 12, 13, 14] through a connection between Hawking radiation and BH QNMs (Bohr-like model for BHs). Let us see this issue in detail. The criticisms by Khriplovich [20] are essentially the following:

1. The exact meaning of Bohr correspondence principle is the following. In quantized systems, the energy jump ΔE between two neighbouring levels with large quantum numbers i. e. between levels with n and $n + 1$, being $n \gg 1$, is related to the classical frequency ω of the system by the formula [20]

$$\Delta E = \omega. \quad (5)$$

Thus, in a semi-classical approximation with $n \gg 1$, the frequencies which corresponds to transitions between energy levels with $\Delta n \ll n$ are integer multiples of the classical frequency ω . Khriplovich concludes by claiming that contrary to the assumption by Hod [4, 5], in the discussed problem of a BH, large quantum numbers n of Bohr correspondence principle are unrelated to the asymptotics (4) of QNMs, but are quantum numbers of the BH itself.

2. The real part QNMs does not differ appreciably from its asymptotic value (4) in the whole numerically investigated range of n , starting from $n \sim 1$. Meanwhile, the imaginary part grows as $n + \frac{1}{2}$, and together with it the spectral width of QNMs (in terms of common frequencies) also increases linearly with n . In this situation, the idea that the resolution of a QNM becomes better and better with the growth of n , and that in the limit $n \rightarrow \infty$ this mode resolves an elementary edge (or site) of a quantized surface, is not reasonable.

4 Improvements by Maggiore

Maggiore [6] showed that the spectrum of BH QNMs can be analysed in terms of superposition of damped oscillations, of the form [6]

$$\exp(-i\omega_I t)[a \sin \omega_R t + b \cos \omega_R t] \quad (6)$$

with a spectrum of complex frequencies $\omega = \omega_R + i\omega_I$. A damped harmonic oscillator $\mu(t)$ is governed by the equation [6]

$$\ddot{\mu} + K\dot{\mu} + \omega_0^2 \mu = F(t), \quad (7)$$

where K is the damping constant, ω_0 the proper frequency of the harmonic oscillator, and $F(t)$ an external force per unit mass. If $F(t) \sim \delta(t)$, i.e. considering the response to a Dirac delta function, the result for $\mu(t)$ is a superposition of a term oscillating as $\exp(i\omega t)$ and of a term oscillating as $\exp(-i\omega t)$, see [6] for details. Then, the behavior (6) is reproduced by a damped harmonic oscillator, through the identifications [6]

$$\frac{K}{2} = \omega_I, \quad \sqrt{\omega_0^2 - \frac{K^2}{4}} = \omega_R, \quad (8)$$

which gives

$$\omega_0 = |\omega| = \sqrt{\omega_R^2 + \omega_I^2}. \quad (9)$$

In [6] it has been emphasized that the identification $\omega_0 = \omega_R$ is correct only in the approximation $\frac{K}{2} \ll \omega_0$, i.e. only for very long-lived modes. For a lot of BH QNMs, for example for highly excited modes, the opposite limit can be correct. Maggiore [6] used this observation to re-examine some aspects of BH quantum physics that were discussed in previous literature assuming that the relevant frequencies were $(\omega_R)_n$ rather than $(\omega_0)_n$. One can indeed easily check

that criticisms in point 2 above have been well addressed by the observation by Maggiore [6], who suggested that one must take

$$\Delta M = \omega = (\omega_0)_n - (\omega_0)_{n-1}, \quad (10)$$

where $(\omega_0)_n \equiv |\omega_n|$, instead of the value (4) proposed in [4, 5]. In fact, the imaginary part becomes dominant for large n and, in turn, the idea that the resolution of a QNM becomes better and better with the growth of n , and that in the limit $n \rightarrow \infty$ this mode resolves an elementary edge (or site) of a quantized surface works. In that way, the minimum quantum which can be absorbed in the transition becomes [6]

$$\Delta M = \omega = \frac{1}{4M}, \quad (11)$$

which permits to find the original Bekensteins result $\Delta A = 8\pi$.

In next Section we will discuss a semi-classical model of quantum BH that will permit to well address point 1 by Khriplovich

5 The Bohr-like model for black holes

Let us return on the connection between BH QNMs and Hawking radiation. Working in strictly thermal approximation, one writes down the probability of emission of Hawking quanta as [21, 22, 23]

$$\Gamma \sim \exp\left(-\frac{\omega}{T_H}\right), \quad (12)$$

being $T_H \equiv \frac{1}{8\pi M}$ the Hawking temperature and ω the energy-frequency of the emitted radiation respectively.

The important correction by Parikh and Wilczek, due to the BH back reaction yields [21, 22]

$$\Gamma \sim \exp\left[-\frac{\omega}{T_H}\left(1 - \frac{\omega}{2M}\right)\right]. \quad (13)$$

This result takes into account the BH varying geometry and adds the term $\frac{\omega}{2M}$ like correction [21, 22]. We have improved the Parikh and Wilczek framework by showing that the probability of emission (13) is indeed associated to the two *non* strictly thermal distributions [24]

$$\langle n \rangle_{boson} = \frac{1}{\exp[4\pi(2M - \omega)\omega] - 1}, \quad \langle n \rangle_{fermion} = \frac{1}{\exp[4\pi(2M - \omega)\omega] + 1}, \quad (14)$$

for bosons and fermions respectively. Equations (14) show an important deviation from the standard Bose-Einstein and Fermi-Dirac distributions which are [24]

$$\langle n \rangle_{boson} = \frac{1}{\exp(8\pi M\omega) - 1} \quad \langle n \rangle_{fermion} = \frac{1}{\exp(8\pi M\omega) + 1}. \quad (15)$$

In fact, the probability of emission of Hawking quanta found by Parikh and Wilczek, i.e. eq. (13), shows that the BH does NOT emit like a perfect black body, i.e. it has not a strictly thermal behavior. On the other hand, the temperature in Bose-Einstein and Fermi-Dirac distributions is a perfect black body temperature. Thus, when we have deviations from the strictly thermal behavior, i.e. from the perfect black body, one expects also deviations from Bose-Einstein and Fermi-Dirac distributions. This is the reason of the difference between eqs. (14) and (15).

It is well known that in various fields of physics and astrophysics the deviation of the spectrum of an emitting body from the strict black body spectrum is taken into account by introducing an *effective temperature*, which represents the temperature of a black body emitting the same total amount of radiation. The effective temperature, which is a frequency dependent quantity, can be introduced in BH physics too [11, 12, 13, 14, 24] as

$$T_E(\omega) \equiv \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)}. \quad (16)$$

Defining $\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}$, one rewrites eq. (13) in Boltzmann-like form as [11, 12, 13, 14, 24]

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp\left(-\frac{\omega}{T_E(\omega)}\right), \quad (17)$$

where one introduces the *effective Boltzmann factor* $\exp[-\beta_E(\omega)\omega]$ appropriate for a BH having an inverse effective temperature $T_E(\omega)$ [11, 12, 13, 14, 24]. Then, the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M - \omega}$ represents the deviation of the BH radiation spectrum from the strictly thermal character [11, 12, 13, 14, 24]. In correspondence of $T_E(\omega)$ one can also introduce the *effective mass* and of the *effective horizon* [11, 12, 13, 14, 24]

$$M_E \equiv M - \frac{\omega}{2}, \quad r_E \equiv 2M_E \quad (18)$$

of the BH *during* the emission of the particle, i.e. *during* the BH contraction phase [11, 12, 13, 14, 24]. Such quantities are average values of the mass and the horizon *before* and *after* the emission [11, 12, 13, 14, 24].

The correction to the thermal spectrum is also very important for the physical interpretation of BH QNMs, and, in turn, is very important to realize the underlying quantum gravity theory as BHs represent theoretical laboratories for developing quantum gravity and BH QNMs are the best candidates like quantum levels [11, 12, 13, 14, 24].

In the appendix of [13] we have rigorously shown that, if one takes into account the deviation from the strictly thermal behavior of the spectrum, eq. (1) must be replaced with

$$\begin{aligned} \omega_n &= \ln 3 \times T_E(\omega_n) + 2\pi i(n + \frac{1}{2}) \times T_E(\omega_n) + \mathcal{O}(n^{-\frac{1}{2}}) = \\ &= \frac{\ln 3}{4\pi[2M - (\omega_0)_n]} + \frac{2\pi i}{4\pi[2M - (\omega_0)_n]}(n + \frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}). \end{aligned} \quad (19)$$

In other words, the Hawking temperature T_H is replaced by the effective temperature T_E in eq. (1). We can also give an intuitive but very enlightening physical

interpretation of such a replacing. Considering the deviation from the thermal spectrum it is natural to assume that the Euclidean BH solution converges now to a *non-thermal* circle at infinity [11, 12]. Therefore, it is straightforward the replacement [11, 12]

$$\beta_H = \frac{1}{T_H} \rightarrow \beta_E(\omega) = \frac{1}{T_E(\omega)}, \quad (20)$$

which takes into account the deviation of the radiation spectrum of a black hole from the strictly thermal feature. In this way, the spacing of the poles in eq. (19) coincides with the spacing [11, 12]

$$2\pi iT_E(\omega) = 2\pi iT_H\left(\frac{2M}{2M - \omega}\right), \quad (21)$$

expected for a *non-thermal* Green's function (a dependence on the frequency is present) [11, 12].

Strictly speaking, eqs. (1) and (19) are corrected only for scalar and gravitational perturbations. On the other hand, for large n eq. (19) is well approximated by (we consider the leading term in the imaginary part of the complex frequencies)

$$\omega_n \simeq \frac{2\pi i n}{4\pi [2M - (\omega_0)_n]}, \quad (22)$$

and we have shown in the appendix of [13] that the behavior (22) also holds for $j = 1$ (vector perturbations). In complete agreement with Bohr's correspondence principle, it is trivial to adapt the analysis in [1] in the sense of the Appendix of [13] and, in turn, to show that the behavior (22) holds if j is a half-integer too. In [11, 12, 13] we have shown that the physical solution of (22) for the absolute values of ω_n is

$$(\omega_0)_n = M - \sqrt{M^2 - \frac{n}{2}}. \quad (23)$$

Now, we clarify how the correspondence between QNMs and Hawking radiation works. One considers a BH original mass M . After an high number of emissions of Hawking quanta and eventual absorptions, because neighboring particles can, in principle, be captured by the BH, the BH is at an excited level $n - 1$ and its mass is $M_{n-1} \equiv M - (\omega_0)_{n-1}$ where $(\omega_0)_{n-1}$, is the absolute value of the frequency of the QNM associated to the excited level $n - 1$. $(\omega_0)_{n-1}$ is interpreted as the total energy emitted at that time. The BH can further emit a Hawking quantum to jump to the subsequent level: $\Delta M_n = (\omega_0)_{n-1} - (\omega_0)_n$. Now, the BH is at an excited level n and the BH mass is

$$\begin{aligned} M_n &\equiv M - (\omega_0)_{n-1} + \Delta M_n = \\ &= M - (\omega_0)_{n-1} + (\omega_0)_{n-1} - (\omega_0)_n = M - (\omega_0)_n. \end{aligned} \quad (24)$$

The BH can, in principle, return to the level $n - 1$ by absorbing an energy $-\Delta M_n = (\omega_0)_n - (\omega_0)_{n-1}$. By using eq. (23) one gets immediately [11, 12, 13]

$$\Delta M = \omega = (\omega_0)_{n-1} - (\omega_0)_n = -f_n(M, n) \quad (25)$$

with [11, 12, 13]

$$f_n(M, n) \equiv \sqrt{M^2 - \frac{1}{2}(n-1)} - \sqrt{M^2 - \frac{n}{2}}. \quad (26)$$

One can easily check that in the very large n limit one gets $f_n(M, n) \rightarrow \frac{1}{4M}$. Thus, by using eq. (3) one gets immediately that in the very large n limit two adjacent QNMs resolve an elementary edge (or site) of a quantized surface $\Delta A \rightarrow 8\pi$, which corresponds to the famous historical result by Bekenstein [15] and this cannot be a coincidence. Then, the quantum levels are equally spaced for both emissions and absorptions being $\frac{1}{4M}$ the jump between two adjacent levels, and this also clearly falsifies the criticism 1. by Khriplovich because in our semiclassical approximation with $n \gg 1$, the frequencies which corresponds to transitions between energy levels with $\Delta n \ll n$ are integer multiples of the classical frequency $\omega = \frac{1}{4M}$.

The BH model that we re-analysed here is somewhat similar to the semi-classical Bohr model of the structure of a hydrogen atom [17, 18]. In our BH model during a quantum jump a discrete amount of energy is indeed radiated and, for large values of the principal quantum number n , the analysis becomes independent of the other quantum numbers. In a certain sense, QNMs represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies represent the energy "shells". In Bohr model [17, 18] electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation $E = hf$, where h is the Planck constant and f the transition frequency. In our BH model, QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels according to eq. (25). The similarity is completed if one note that the interpretation of eq. (23) is of a particle, the "electron", quantized with anti-periodic boundary conditions on a circle of length [11, 12, 13]

$$L = \frac{1}{T_E(E_n)} = 4\pi \left(M + \sqrt{M^2 - \frac{n}{2}} \right), \quad (27)$$

which is the analogous of the electron travelling in circular orbits around the hydrogen nucleus, similar in structure to the solar system, of Bohr model [17, 18]. Clearly, all these similarities with the Bohr semi-classical model of the hydrogen atom and all these consistences with well known results in the literature of BHs, starting by the universal Bekenstein's result, *cannot* be coincidences, but are confirmations of the correctness of the current analysis.

It is also important to stress that the Bohr-like BH has important implications for the BH information paradox [25]. In fact, the BH seems to be a well defined quantum mechanical system, having an ordered, discrete quantum spectrum. This is surely consistent with the unitarity of the underlying quantum

gravity theory and with the idea that information should come out in BH evaporation. We have indeed recently shown that the time evolution of the Bohr-like BH obeys a *time dependent Schrödinger equation* [26]. In that way, the physical state and the correspondent *wave function* are written in terms of an *unitary* evolution matrix instead of a density matrix [26]. Thus, the final state results to be a *pure* quantum state instead of a mixed one while the entanglement problem connected with the information paradox is solved showing that the emitted radiation is entangled with BH QNMs [26].

6 Conclusion remarks

Khriplovich [20] verbatim claimed that “the correspondence principle does not dictate any relation between the asymptotics of quasinormal modes and the spectrum of quantized black holes” and that “this belief is in conflict with simple physical arguments”. In this paper we have shown that the criticisms in [20] work only for the original proposal by Hod, while they do not work for the improvements suggested by Maggiore and recently finalized by the author and collaborators through a connection between Hawking radiation and BH QNMs. Thus, QNMs can be really interpreted as BH quantum levels in a Bohr-like model for BHs where QNMs represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies represent the energy "shells". Then, the emission of Hawking quanta and the absorptions of particles represent the jumps among the various quantum levels.

The results in this paper have also important consequences on the BH information paradox.

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